Q1.

Define binary variable $x_i = \begin{cases} 1, & \text{if } s_i \text{ is evaluated} \\ 0, & \text{otherwise} \end{cases}$

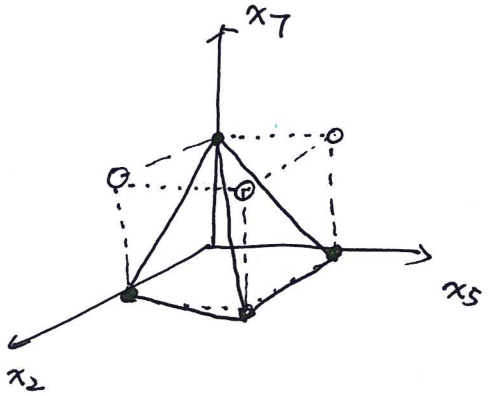
(i) if $x_1 = 1$, and $x_4 = 1$, then $x_6 = 0$.

$\Rightarrow$ $(x_1 = 1,\ x_4 = 1,\ x_6 = 1)$ is excluded.

$\Rightarrow$ $x_1 + x_4 + x_6 \le 2$.

(ii) if $x_2 = 1$ or $x_5 = 1$, then $x_7 = 0$.

forbidden : $\begin{cases} x_2 = 1, & x_5 = 0, & x_7 = 1 \\ x_2 = 0, & x_5 = 1, & x_7 = 1 \\ x_2 = 1, & x_5 = 1, & x_7 = 1 \end{cases}$



$\Rightarrow$ $\begin{cases} x_2 + x_7 \le 1 \\ x_5 + x_7 \le 1 \end{cases}$

(iii) only two and exactly two sites are evaluated of the group - S_3, S_4, S_5, S_6.

$\Rightarrow$ $x_3 + x_4 + x_5 + x_6 = 2$.

(iv) Either (site 4 ~~and~~ site 6) or (site 7 and 8) are evaluated, but not both.

$\Rightarrow$ $x_4 + x_6 = 2$ or $x_7 + x_8 = 2$.

$\Rightarrow$ $\begin{cases} x_4 + x_6 = 2y \\ x_7 + x_8 = 2(1-y) \end{cases}$, y binary

$\Rightarrow$ $\min \sum_{i=1}^{10} c_i x_i$

s.t. $x_1 + x_4 + x_6 \le 2$

$x_2 + x_7 \le 1$

$x_5 + x_7 \le 1$

$x_3 + x_4 + x_5 + x_6 = 2$

$x_4 + x_6 = 2y$. $x_7 + x_8 \ge 2(1-y)$

$\sum_{i=1}^{10} x_i \ge 5$, x_i. $y = 0$ or 1. $i = 1, \cdots 10$.

Q2.

(a) (1st row) $\frac{2}{5}x_3 + \frac{4}{5}x_4 \geq \frac{1}{5}$

(2nd row) $\frac{1}{5}x_3 + \frac{2}{5}x_4 \geq \frac{3}{5}$

(b) 1st cut:

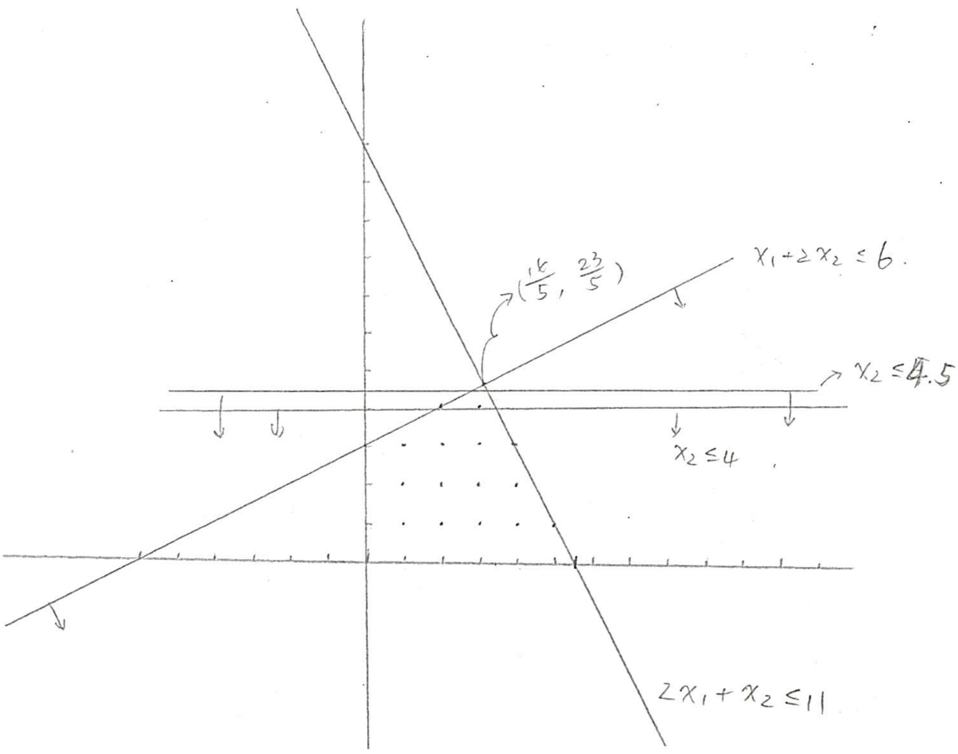
$$\frac{2}{5}(11-2x_1-x_2) + \frac{4}{5}(6+x_1-2x_2) \geq \frac{1}{5}$$

$$\Rightarrow \quad x_2 \leq 4.5$$

2nd cut:

$$\frac{1}{5}(11-2x_1-x_2) + \frac{2}{5}(6+x_1-2x_2) \geq \frac{3}{5}$$

$$\Rightarrow \quad x_2 \leq 4.$$



(C) Add the 2nd cut

$$\frac{1}{5}x_3 + \frac{2}{5}x_4 \geq \frac{3}{5}.$$

$$\Rightarrow -\frac{1}{5}x_3 - \frac{2}{5}x_4 + x_5 = -\frac{3}{5}$$

	x_1	x_2	x_3	x_4	x_5	b
x_1	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{16}{5}$
x_2	0	1	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{23}{5}$
x_5	0	0	$-\frac{1}{5}$	$-\frac{2}{5}^*$	1	$-\frac{3}{5}$ ←
z	0	0	$\frac{11}{5}$	$\frac{2}{5}$	0	$\frac{133}{5}$

	x_1	x_2	x_3	x_4	x_5	b
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{7}{2}$
x_2	0	1	0	0	1	4
x_4	0	0	2	1	$-\frac{5}{2}$	$\frac{3}{2}$
z	0	0	2	0	1	26

Still not feasible.

Generate a new cut from the first row.

$$\frac{1}{2}x_3 + \frac{1}{2}x_5 \geq \frac{1}{2} \Rightarrow x_3 + x_5 \geq 1$$

$$-x_3 - x_5 + x_6 = -1$$

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{7}{2}$
x_2	0	1	0	0	1	0	4
x_4	0	0	2	1	$-\frac{5}{2}$	0	$\frac{3}{2}$
x_6	0	0	-1	0	-1^*	1	-1 ←
z	0	0	2	0	1	0	26

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_1	1	0	1	0	0	$-\frac{1}{2}$	4
x_2	0	1	-1	0	0	1	3
x_4	0	0	$\frac{9}{2}$	1	0	$-\frac{5}{2}$	4
x_5	0	0	1	0	1	-1	1
	0	0	1	0	0	1	25

$\Rightarrow$ optimal solution $x_1 = 4$ $x_2 = 3$. $x_4 = 4$ and $x_5 = 1$

with $z^* = 25$.

Q3.

(a) LP-relaxation.

$$\text{Max} \quad z = 50x_1 + 60x_2 + 140x_3 + 40x_4$$

$$\text{s.t.} \quad 5x_1 + 10x_2 + 20x_3 + 20x_4 \le 30$$

$$0 \le x_i \le 1 \quad i = 1, 2, 3, 4.$$

Greedy algorithm to find an optimal solution to the LP-relaxation.

	item	1	2	3	4
$\frac{\text{value}}{\text{weight cost}}$	= unit value	10	6	7	2

$\Rightarrow$ optimal solution to LP-relaxation.

$(x_1, x_2, x_3, x_4) = (1, 0.5, 1, 0)$

With $z = 50 + 30 + 140 = 220$.

(b) We use the optimal solution to ~~PE~~ LP-relaxation above as the upper bound 220 for the maximum objective value. Since x_2 is not an integer, we branch on x_2 and get 2 branches, one where we set $x_2 = 1$ and one where $x_2 = 0$.

As suggested in the question, we first treat the $x_2 = 1$ branch. The remaining capacity is 20. So the optimal solution to the resulting LP-relaxation of the knapsack problem is

LP(2)
$$\text{Max} \quad 50x_1 + 60x_2 + 140x_3 + 40x_4$$

$$\text{s.t.} \quad 5x_1 + 10x_2 + 20x_3 + 20x_4 \le 30$$

$$0 \le x_1, x_2, x_3, x_4 \le 1, \quad x_2 = 1$$

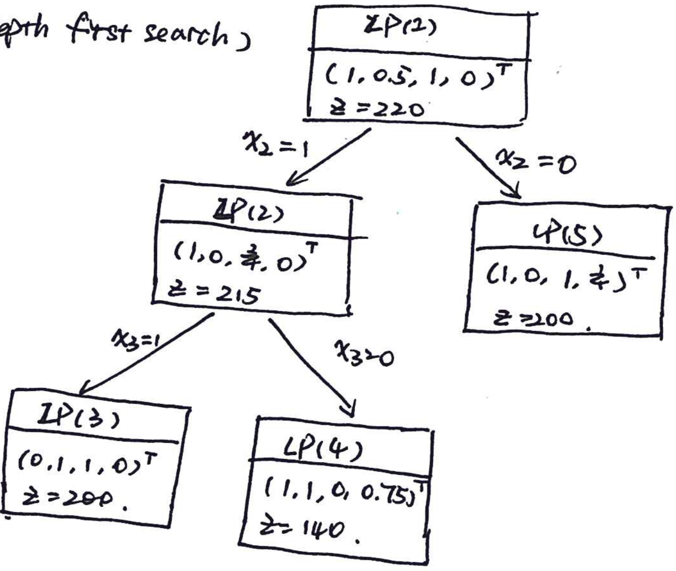
$\Rightarrow$ optimal solution $\vec{x} = (1, 0, \frac{3}{4}, 0)^T$ with $z^* = 215$.

As x_3 is fractional, we branch on that and first consider $x_3 = 1$ (and $x_2 = 1$) (Depth first search)

LP(3) Max $50x_1 + 60 + 140 + 40x_4$

s.t. $5x_1 + 10 + 20 + 20x_4 \leq 30$

$0 \leq x_1, x_4 \leq 1$



⇒ optimal solution

$\vec{x} = (0, 1, 1, 0)^T$ with $z^* = 200$

⇒ fathom this node by finding a feasible solution.

Now, consider $x_3 = 0$ (and $x_2 = 1$).

LP(4) Max $50x_1 + 60 + 40x_4$

s.t. $5x_1 + 10 + 20x_4 \leq 30$

$0 \leq x_1, x_4 \leq 1$

⇒ optimal solution

$\vec{x} = (1, 1, 0, 0.75)^T$ with $z = 140$.

⇒ fathom this node by bounding argument.

Now consider the branch $x_2 = 0$.

LP(5) Max $50x_1 + 140x_3 + 40x_4$

s.t. $5x_1 + 20x_3 + 20x_4 \leq 30$

$0 \leq x_1, x_3, x_4 \leq 1$

⇒ optimal solution

$\vec{x} = (1, 0, 1, \frac{1}{4})^T$ with $z^* = 200$.

⇒ fathom this node by bounding argument.

⇒ we stop the whole search.

⇒ optimal solution is hence $x_2 = 1$ and $x_3 = 1$. $x_1 = x_4 = 0$ with $z = 200$.

Q4 (We can solve it using Resource Allocation Model).

optimal value function:

$f_k(x)$ = maximum grade points to be obtained from course k to 4 with x days available to allocate.

recurrence relation:

$$f_k(x) = \max_{y=1,2,\ldots,\min(x,4)} [r_k(y) + f_{k+1}(x-y)]$$

Boundary condition: $f_4(x) = r_4(x)$

optimal policy function: $P_k(x)$ keeps a record of the optimal allocation of days to course k, if there are x days available.

Solution: $f_1(7)$

k \ x	1	2	3	4	5	6	7
4	6	7	9	9			
3		8 (1)	10 (2)	13 (3)	14 (3 or 4)		
2			13 (1)	15 (1)	18 (1)	19 (1)	
1							23 (2)

$\Rightarrow$ 2 days to course 1.

1 day to course 2

3 days to course 3

1 day to course 4.

Q.4 (Equipment replacement Problem)

$f_k(x)$ = min attainable cost through year k to 4. if owning a machine of age x at the start of year k.

$$f_k(x) = \min \begin{bmatrix} \text{keep}: & c(x) + f_{k+1}(x+1) \\ \text{buy}: & p(k) + c(0) - t(x) + f_{k+1}(1) \end{bmatrix}$$

$x = 1, 2, \cdots k-1$.

boundary condition: $f_5(x) = -t(x)$.

optimal solution: $f_1(0)$.

operating cost

$c(0) = 120$, $c(1) = 144$, $c(2) = 172.8$, $c(3) = 207.36$

resale value

$t(1) = 150$, $t(2) = 135$, $t(3) = 121.5$, $t(4) = 109.35$

$p(k)$ = price of a new mower at the start of year k.

$P(1) = 200$, $P(2) = 220$, $P(3) = 242$, $P(4) = 266.2$

$f_5(x)$.

x	$f_5(x)$
1	−150
2	−135
3	−121.5
4	−109.35.

$f_4(x)$. $x = 1, 2, 3$.

$$f_4(1) = \min\begin{bmatrix} c(1) + f_5(2) \\ P(4) + c(0) - t(1) + f_5(1) \end{bmatrix} = \min\begin{bmatrix} 144 + -135 \\ 266.2 + 120 - 150 + -150 \end{bmatrix} = \min\begin{bmatrix} 9 \\ 86.2 \end{bmatrix} = 9 \quad P_4(1) = K.$$

$$f_4(2) = \min\begin{bmatrix} c(2) + f_5(3) \\ P(4) + c(0) - t(2) + f_5(1) \end{bmatrix} = \min\begin{bmatrix} 172.8 + (-121.5) \\ 266.2 + 120 - 135 + (-150) \end{bmatrix} = \min\begin{bmatrix} 51.3 \\ 101.2 \end{bmatrix} = 51.3 \quad P_4(2) = K$$

$$f_4(3) = \min\begin{bmatrix} c(3) + f_5(4) \\ P(4) + c(0) - t(3) + f_5(1) \end{bmatrix} = \min\begin{bmatrix} 207.36 - 109.35 \\ 266.2 + 120 - 121.5 + (-150) \end{bmatrix} = \min\begin{bmatrix} 98.01 \\ 114.7 \end{bmatrix} = 98.01 \quad P_4(3) = K$$

$f_3(x)$ $x = 1, 2$

$$f_3(1) = \min\begin{bmatrix} c(1) + f_4(2) \\ P(3) + c(0) - t(1) + f_4(1) \end{bmatrix} = \min\begin{bmatrix} 144 + 51.3 \\ 242 + 120 - 150 + 9 \end{bmatrix} = \min\begin{bmatrix} 195.3 \\ 221 \end{bmatrix} = 195.3 \quad P_3(1) = K$$

$$f_3(2) = \min\begin{bmatrix} c(2) + f_4(3) \\ P(3) + c(0) - t(2) + f_4(1) \end{bmatrix} = \min\begin{bmatrix} 172.8 + 98.01 \\ 242 + 120 - 135 + 9 \end{bmatrix} = \min\begin{bmatrix} 270.81 \\ 236 \end{bmatrix} = 236 \quad P_3(2) = B.$$

$f_2(x)$ $x = 1$.

$$f_2(1) = \min\begin{bmatrix} c(1) + f_3(2) \\ P(2) + c(0) - t(1) + f_3(1) \end{bmatrix} = \min\begin{bmatrix} 144 + 236 \\ 220 + 120 - 150 + 195.3 \end{bmatrix} = \min\begin{bmatrix} 380 \\ 385.3 \end{bmatrix} = 380 \quad P_2(1) = K$$

$$f_1(0) = \min\left[c(0) + f_2(1) + P(1) \right] = \cancel{144}\ 120 + 380 + 200 = \cancel{724}\ 700. \quad P_1(0) = K.$$

⇒ optimal policy: KKBK.

Billy's decision is justified.